The classification of root systems

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The elements of R are called *roots*.
The *rank* of the root system is the dimension of $\mathbb{E}$.

Restrictions

Projection

$$\mathrm{proj}_\alpha \beta = \alpha \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \frac{1}{2} n_{\beta\alpha} \alpha$$

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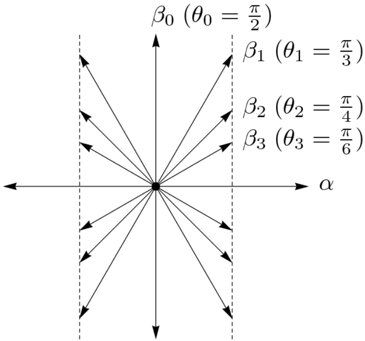
Angles

$$n_{\beta\alpha} = 2 \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 2 \frac{\|\beta\| \, \|\alpha\| \cos\theta}{\|\alpha\|^2} = 2 \frac{\|\beta\|}{\|\alpha\|} \cos\theta \in \mathbb{Z}$$

$$n_{\beta\alpha} \cdot n_{\alpha\beta} = 4 \cos^2 \theta \in \mathbb{Z}$$

$$4 \cos^2 \theta \in \{0, 1, 2, 3, 4\}$$

Geometry

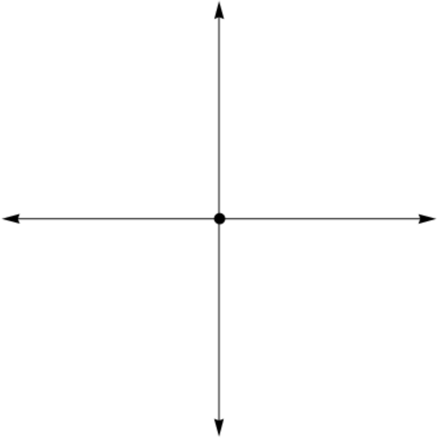


Angles

$$4\cos^2\theta \in \{0, 1, 2, 3\}, \text{ or } \cos\theta \in \pm\left\{0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right\}$$

Examples in rank 2

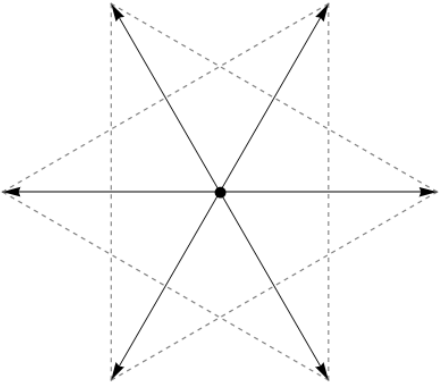
Root system $A_1 \times A_1$



(decomposable)

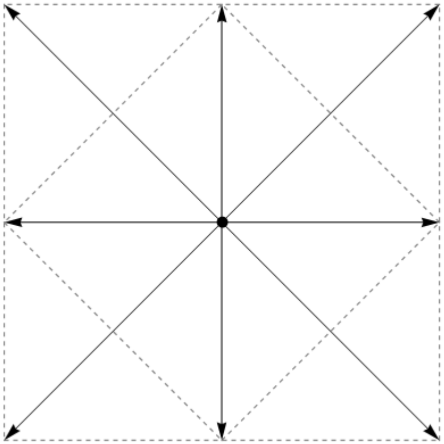
Examples in rank 2

Root system A_2



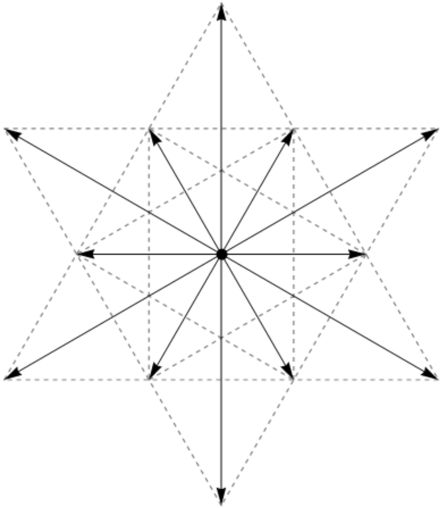
Examples in rank 2

Root system B_2



Examples in rank 2

Root system G_2



Positive roots and simple roots

Consider a vector d, such that $\forall \alpha \in R : \langle \alpha, d \rangle \neq 0$. Define $R^+(d) = \{\alpha \in R | \langle \alpha, d \rangle > 0\}$. Then $R = R^+(d) \cup R^-(d)$, where $R^-(d) = -R^+(d)$.

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Definition

The set of all simple roots of a root system R is called *basis* of R.

Properties of simple roots

Definition

The hyperplanes orthogonal to $\alpha \in R$ cut the space $\mathbb{E}$ into open, connected regions called *Weyl chambers*.

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Lemma
The root system R can be uniquely reconstructed from its basis.

Coxeter and Dynkin diagrams

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If α and β are distinct simple roots, then $\langle \alpha, \beta \rangle \leq 0$.

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Conclusion

Since $4\cos^2\theta \in \{0, 1, 2, 3\}$, it means that $\theta \in \left\{\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}\right\}$.

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Definition

The *Coxeter graph* of a root system R is a graph that has one vertex for each simple root of R and every pair α, β of distinct vertices is connected by $n_{\alpha\beta} \cdot n_{\beta\alpha} = 4\cos^2\theta \in \{0, 1, 2, 3\}$ edges.

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Definition

The *Dynkin diagram* of a root system is its Coxeter graph with arrow attached to each double and triple edge pointing from longer root to shorter root.

Admissible diagrams

Definition

A set of n unit vectors $\{v_1, v_2, \ldots, v_n\} \subset \mathbb{E}$ is called an *admissible configuration* if:

1. v_i's are linearly independent and span $\mathbb{E}$,
2. if $i \neq j$, then $\langle v_i, v_j \rangle \leq 0$,
3. and $4 \langle v_i, v_j \rangle^2 = 4 \cos^2 \theta \in \{0, 1, 2, 3\}$.

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Note

The set of normalized simple roots of any root system is an admissible configuration (they are linearly independent, span the whole space, and have specific angles between them).

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Coxeter graph of an admissible configuration is *admissible diagram*.

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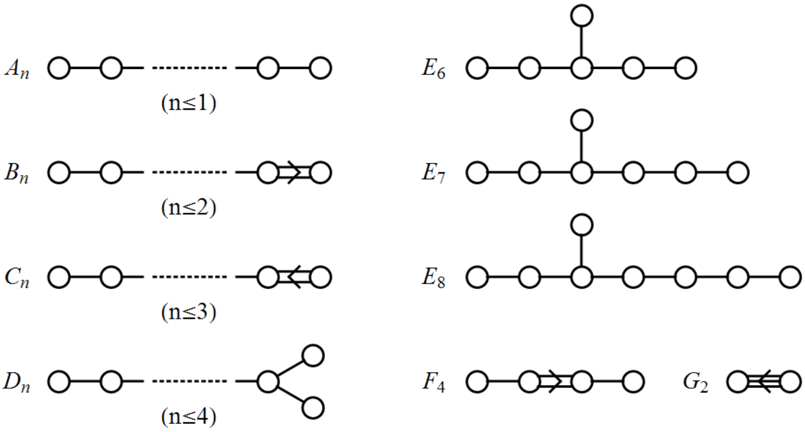
Conclusion

It means, the set of simple roots of an irreducible root system can not be decomposed into mutually orthogonal subsets. Hence the corresponding Coxeter graph will be *connected*. Thus, to classify all irreducible root systems, it is enough to consider only connected admissible diagrams.

Classification theorem

Theorem

The Dynkin diagram of an irreducible root system is one of:



Step 1

Claim: *Any subdiagram of an admissible diagram is also admissible.*

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If the set $\{v_1, v_2, \ldots, v_n\}$ is an admissible configuration, then clearly any subset of it is also an admissible configuration (in the space it spans). The same holds for admissible diagrams.

Step 2

Claim: *A connected admissible diagram is a tree.*

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Define $v = \sum_{i=1}^{n} v_i$ ($v \neq 0$). Then

$$0 < \langle v, v \rangle = \sum_{i=1}^{n} \langle v_i, v_i \rangle + \sum_{i<j} 2 \langle v_i, v_j \rangle = n + \sum_{i<j} 2 \langle v_i, v_j \rangle .$$

If v_i and v_j are connected, then

$$2 \langle v_i, v_j \rangle \in \left\{ -1, -\sqrt{2}, -\sqrt{3} \right\}$$

In particular, $2 \langle v_i, v_j \rangle \leq -1$. It means, the number of terms in the sum and hence the number of edges can not exceed $n - 1$.

Step 3

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Let $v_1, v_2, \ldots, v_k$ be connected to c, then $\langle v_i, v_j \rangle = \delta_{ij}$. Let $v_0 \neq 0$ be the normalized projection of c to the orthogonal complement of v_i's. Then $\{v_0, v_1, v_2, \ldots, v_k\}$ is an orthonormal basis and:

$$c = \sum_{i=0}^{k} \langle c, v_i \rangle v_i.$$

Since $\langle c, c \rangle = \sum_{i=0}^{k} \langle c, v_i \rangle^2 = 1$ and $\langle c, v_0 \rangle \neq 0$, then

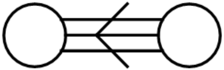
$$\sum_{i=1}^{k} 4 \langle c, v_i \rangle^2 < 4,$$

where $4 \langle c, v_i \rangle^2$ is the number of edges between c and v_i.

Step 4

Claim: *The only connected admissible diagram containing a triple edge is*

G_2



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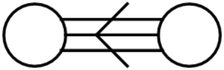

This follows from the previous step. From now on we will consider only diagrams with single and double edges.

Step 5

Claim: *Any simple chain* $v_1, v_2, \ldots, v_k$ *can be replaced by a single vector* $v = \sum_{i=1}^{k} v_i$.

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Vector v is a unit vector, since $2\langle v_i, v_j\rangle = -\delta_{i+1,j}$ and therefore

$$\langle v, v\rangle = k + \sum_{i<j} 2\langle v_i, v_j\rangle = k + \sum_{i=1}^{k-1} 2\langle v_i, v_{i+1}\rangle = k - (k-1) = 1.$$

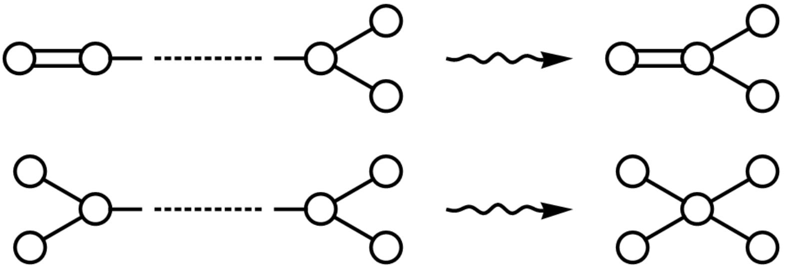
If u is not in the chain, then it can be connected to at most one vertex in the chain (let it be v_j). Then

$$\langle u, v\rangle = \sum_{i=1}^{k} \langle u, v_i\rangle = \langle u, v_j\rangle$$

and u remains connected to v in the same way. Therefore the obtained diagram is also admissible and connected.

Step 6

Claim: *A connected admissible diagram has none of the following subdiagrams:*



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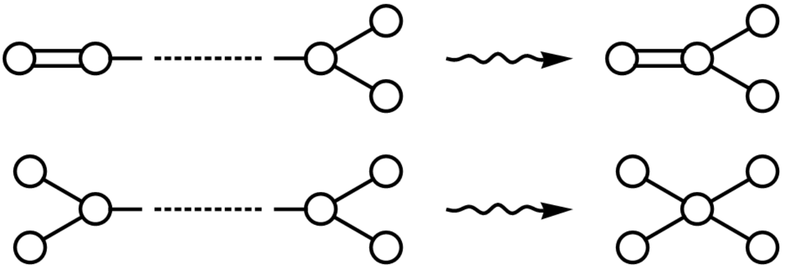


Conclusion

It means that a connected admissible diagram can contain at most one double edge and at most one branching, but not both of them simultaneously.

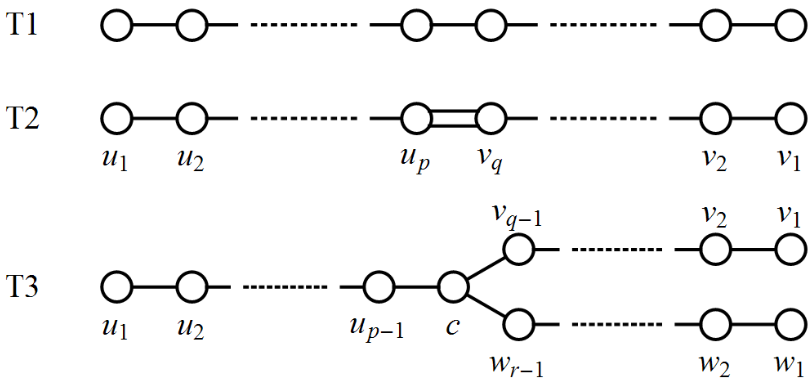
Step 7

Claim: *There are only three types of connected admissible diagrams:*

T1: *a simple chain,*

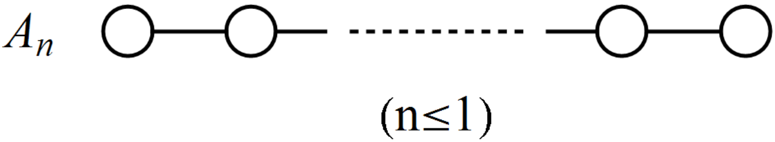
T2: *a diagram with a double edge,*

T3: *a diagram with branching.*



Step 8

Claim: *The admissible diagram of type T1 corresponds to the Dynkin diagram A_n, where $n \geq 1$.*



Step 9

Claim: *The admissible diagrams of type T2 are* F_4, B_n, *and* C_n.

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Define $u = \sum_{i=1}^{p} i \cdot u_i$. Since $2\langle u_i, u_{i+1}\rangle = -1$ for $1 \leq i \leq p-1$,

$$\langle u, u\rangle = \sum_{i=1}^{p} i^2 \langle u_i, u_i\rangle + \sum_{i<j} ij \cdot 2\langle u_i, u_j\rangle = \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1)$$

$$= p^2 - \sum_{i=1}^{p-1} i = p^2 - \frac{p(p-1)}{2} = \frac{p(p+1)}{2}.$$

Similarly, $v = \sum_{j=1}^{q} j \cdot v_j$ and $\langle v, v\rangle = q(q+1)/2$. From $\langle u, v\rangle = pq\langle u_p, v_q\rangle$ and $4\langle u_p, v_q\rangle^2 = 2$ we get $\langle u, v\rangle^2 = p^2q^2/2$. From Cauchy-Schwarz inequality $\langle u, v\rangle^2 < \langle u, u\rangle \langle v, v\rangle$ we get

$$\frac{p^2q^2}{2} < \frac{p(p+1)}{2} \cdot \frac{q(q+1)}{2}.$$

Step 10 (continued)

Since $p, q \in \mathbb{Z}_+$, we get $2pq < (p+1)(q+1)$ or simply $(p-1)(q-1) < 2$.

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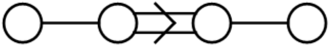
F_4



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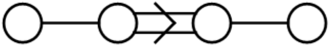
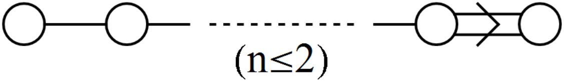
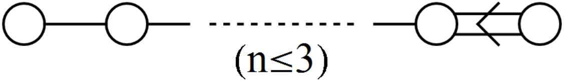


$p = 1$ and q is arbitrary (or vice versa)

B_n



C_n



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Define $u = \sum_{i=1}^{p-1} i \cdot u_i$, $v = \sum_{j=1}^{q-1} j \cdot v_j$, and $w = \sum_{k=1}^{r-1} k \cdot w_k$. Let u', v', and w' be the corresponding unit vectors. Then

$$1 = \langle c, c \rangle > \left\langle c, u' \right\rangle^2 + \left\langle c, v' \right\rangle^2 + \left\langle c, w' \right\rangle^2 .$$

Since $\langle c, u_i \rangle^2 = 0$ unless $i = p-1$ and $4 \langle c, u_{p-1} \rangle^2 = 1$, we have

$$\langle c, u \rangle^2 = \sum_{i=1}^{p-1} i^2 \langle c, u_i \rangle^2 = (p-1)^2 \langle c, u_{p-1} \rangle^2 = \frac{(p-1)^2}{4}.$$

We already know that $\langle u, u \rangle = p(p-1)/2$, therefore

$$\left\langle c, u' \right\rangle^2 = \frac{\langle c, u \rangle^2}{\langle u, u \rangle} = \frac{(p-1)^2}{4} \cdot \frac{2}{p(p-1)} = \frac{p-1}{2p} = \frac{1}{2}\left(1 - \frac{1}{p}\right).$$

Step 10 (Continued)

If we do the same for v and w, we get
$2 > (1 - 1/p) + (1 - 1/q) + (1 - 1/r)$ or simply

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1, \quad p, q, r \geq 2.$$

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We can assume that $p \geq q \geq r \geq 2$. There is no solution with $r \geq 3$, since then the sum can not exceed 1. Therefore we have to take $r = 2$. If we take $q = 2$ as well, then any p suits, but for $q = 3$ we have $1/q + 1/r = 5/6$ and we can take only $p < 6$. There are no solutions with $q \geq 4$, because then the sum is at most 1.

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p	q	r	Dynkin diagram
any	2	2	D_n
3	3	2	E_6
4	3	2	E_7
5	3	2	E_8

End of proof

Q.E.D.

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Theorem

For each Dynkin diagram we have found there indeed is an irreducible root system having the given diagram.